A unified approach to spinors in Hilbert space

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Abstract

To a complex Hilbert space V one may associate a C*-algebra called the "canonical anti-commutation relation" algebra $\mathrm{CAR}(V)$. This algebra is, loosely speaking, the C*-algebra generated by V, such that $vw+wv=0$ for all $v,w \in V$, and such that unit vectors in V become unitary elements in $\mathrm{CAR}(V)$. Alternatively, one can consider the complex Hilbert space $V \oplus V^*$, which comes equipped with a canonical real structure. To such a Hilbert space equipped with a real structure one may associate a Clifford algebra, $\mathrm{Cl}(V \oplus V^*)$. The CAR algebra $\mathrm{CAR}(V)$ and the Clifford algebra $\mathrm{Cl}(V)$ are well-studied objects that are widely understood to be "the same". The goal of this note is to make precise in which way they are the same.

1 Introduction

I make no claim to any originality in these notes; their purpose is to summarize some observations that are probably well-known, but perhaps not easy to find in the literature. Things that are easy to find are explained only summarily. An excellent account of infinite-dimensional Clifford algebras and their representations can be found in [PR94]. For a definition of the CAR-algebra, a starting point might be [Ten17].

2 The CAR algebra and the Clifford algebra

Let V be a complex Hilbert space.

Definition 2.1. If A is a unital C*-algebra and $f : V \to A$ is a map, then f is a *CAR-map* if f is linear, and

$$f(v)f(w) + f(w)f(v) = 0,$$
$$f(v)f(w)^* + f(w)^*f(v) = \langle v, w\rangle \mathbb{1}_A,$$

for all $v, w \in V$. △

Definition 2.2. The *canonical anti-commutation relation* (CAR) *algebra*, $\mathrm{CAR}(V)$, is the universal unital C*-algebra with respect to CAR-maps $f : V \to A$. In other words, $\mathrm{CAR}(V)$ comes equipped with an injective CAR-map $a_V : V \to \mathrm{CAR}(V)$, such that if $f : V \to A$ is any other CAR-map, then there exists a unique $*$-homomorphism $\tilde{f} : \mathrm{CAR}(V) \to A$ such that $f = \tilde{f} \circ a_V$. △

It is common to suppress the inclusion map $a_V : V \to \mathrm{CAR}(V)$, we will do so only when there is no risk of confusion.

Definition 2.3. A *real structure* on a complex Hilbert space H is a conjugate linear isometric involution $\alpha : H \to H$. △

We write V^* for the complex-linear dual of V, and we write $\iota : V \to V^*$ for the conjugate-linear isometric isomorphism induced by the inner product of V. We write $\widehat{V}$ for the complex Hilbert space $V \oplus V^*$, equipped with the real structure

$$\alpha_V = \begin{pmatrix} 0 & \iota^* \\ \iota & 0 \end{pmatrix}.$$

Let H be a complex Hilbert space equipped with a real structure α.

Remark 2.4. Throughout, our convention will be that V is a complex Hilbert space, without a given real structure; while H is a complex Hilbert space with a given real structure α. If we write $\widehat{H}$, then we mean the complex Hilbert space $H \oplus H^*$ with the real structure α_H; the original real structure α is not involved.

Moreover, our inner products are complex linear in the first entry, and conjugate linear in the second. △

Definition 2.5. If A is a unital C*-algebra and $f : H \to A$ is a map, then f is a *Clifford map* if f is linear, and

$$f(v)f(w) + f(w)f(v) = \langle v, \alpha(w)\rangle \mathbb{1}_A, \qquad\qquad f(\alpha(v)) = f(v)^*,$$

for all $v, w \in V$. △

Definition 2.6. The *Clifford algebra*, $\mathrm{Cl}(H)$, is the universal unital C*-algebra with respect to Clifford-maps $f : H \to A$. In other words, $\mathrm{Cl}(H)$ comes equipped with an injective Clifford map $i_H : H \to \mathrm{Cl}(H)$, such that if $f : H \to A$ is any other Clifford map, then there exists a unique $*$-homomorphism $\tilde{f} : \mathrm{Cl}(H) \to A$ such that $f = \tilde{f} \circ i_H$. △

Again, it is common to suppress the inclusion map $i_H : H \to \mathrm{Cl}(H)$, we will do so only when there is no risk of confusion.

Proposition 2.7. *The inclusion map $j : V \to \widehat{V}$ extends to an isomorphism of C*-algebras $u : \mathrm{CAR}(V) \to \mathrm{Cl}(\widehat{V})$.*

Proof. First, observe that the map $V \xrightarrow{j} \widehat{V} \xrightarrow{i_{\widehat{V}}} \mathrm{Cl}(\widehat{V})$ is a CAR-map.

$$\begin{aligned} i_{\widehat{V}}j(v)i_{\widehat{V}}j(w) + i_{\widehat{V}}j(w)i_{\widehat{V}}j(v) &= i_{\widehat{V}}(v,0)i_{\widehat{V}}(w,0) + i_{\widehat{V}}(w,0)i_{\widehat{V}}(v,0) \\ &= \langle (v,0), \alpha_V(w,0)\rangle \\ &= \langle (v,0), (0,\iota(w))\rangle \mathbb{1} = 0, \end{aligned}$$

$$\begin{aligned} i_{\widehat{V}}j(v)i_{\widehat{V}}j(w)^* + i_{\widehat{V}}j(w)^*i_{\widehat{V}}j(v) &= i_{\widehat{V}}(v,0)i_{\widehat{V}}(w,0)^* + i_{\widehat{V}}(w,0)^*i_{\widehat{V}}(v,0) \\ &= i_{\widehat{V}}(v,0)i(0,\iota(w)) + i_{\widehat{V}}(0,\iota(w))i_{\widehat{V}}(v,0) \\ &= \langle (v,0),(w,0)\rangle \mathbb{1} = \langle v,w\rangle \mathbb{1}. \end{aligned}$$

We thus obtain a $*$-homomorphism $\tilde{j} : \mathrm{CAR}(V) \to \mathrm{Cl}(\widehat{V})$ such that $\tilde{j}a_V = i_{\widehat{V}}j$. Let $q : \widehat{V} \to \mathrm{CAR}(V)$ be the map

$$q(v,\iota(w)) = a_V(v) + a_V(w)^*.$$

The map q is a Clifford map, and thus extends to a $*$-homomorphism $\tilde{q} : \mathrm{Cl}(\widehat{V}) \to \mathrm{CAR}(V)$.

$$\begin{aligned} q(v,\iota(w))q(x,\iota(y)) + q(x,\iota(y))q(v,\iota(w)) &= (a_V(v) + a_V(w)^*)(a_V(x) + a_V(y)^*) \\ &\quad + (a_V(x) + a_V(y)^*)(a_V(v) + a_V(w)^*) \\ &= a_V(v)a_V(x) + a_V(v)a_V(y)^* + a_V(w)^*a_V(x) + a_V(w)^*a_V(y)^* \\ &\quad + a_V(x)a_V(v) + a_V(x)a_V(w)^* + a_V(y)^*a_V(v) + a_V(y)^*a_V(w)^* \\ &= a_V(v)a_V(y)^* + a_V(w)^*a_V(x) + a_V(x)a_V(w)^* + a_V(y)^*a_V(v) \\ &= \langle v,y\rangle \mathbb{1} + \langle x,w\rangle \mathbb{1} \\ &= \langle v,y\rangle \mathbb{1} + \langle \iota(w),\iota(x)\rangle \mathbb{1} \\ &= \langle (v,\iota(w)),(y,\iota(x))\rangle \mathbb{1} \\ &= \langle (v,\iota(w)),\alpha(x,\iota(y))\rangle \mathbb{1} \end{aligned}$$

$$q(v,\iota(w))^* = (a_V(v) + a_V(w)^*)^* = a_V(w) + a_V(v)^* = q(w,\iota(v)) = q(\alpha(v,\iota(w))).$$

We claim that $\tilde{q}\tilde{j} = \mathbb{1}$ and $\tilde{j}\tilde{q} = \mathbb{1}$. First, let $v \in V$ be arbitrary. We then compute

$$\tilde{q}\tilde{j}a_V(v) = \tilde{q}i_{\widehat{V}}j(v) = qj(v) = a_V(v)$$

In other words, the map $\tilde{q}\tilde{j}$ is the extension of the map $a_V : V \to \mathrm{CAR}(V)$ to $\mathrm{CAR}(V)$, i.e. $\tilde{q}\tilde{j} = \mathbb{1}$. Now, let $(v, \iota(w)) \in \widehat{V}$ be arbitrary. We compute

$$\begin{aligned}\tilde{j}\tilde{q}i_{\widehat{V}}(v, \iota(w)) &= \tilde{j}q(v, \iota(w)) = \tilde{j}(a_V(v) + a_V(w)^*) = \tilde{j}(a_V(v)) + \tilde{j}(a_V(w))^* \\ &= i_{\widehat{V}}j(v) + i_{\widehat{V}}j(w)^* = i_{\widehat{V}}(v, 0) + i_{\widehat{V}}(0, \iota(w)) = i_{\widehat{V}}(v, \iota(w)).\end{aligned}$$

Thus, $\tilde{j}\tilde{q}$ extends the map $i_{\widehat{V}} : \widehat{V} \to \mathrm{Cl}(\widehat{V})$ to $\mathrm{Cl}(\widehat{V})$, i.e. $\tilde{j}\tilde{q} = \mathbb{1}$. □

3 The Fock representations

Given a Hilbert space V, we write ΛV for the Hilbert space completion of the exterior algebra of V, i.e.

$$\Lambda V := \left(\oplus_{k=0}^{\infty} \wedge^k V\right)^{\langle\cdot,\cdot\rangle}.$$

Definition 3.1. A *polarization* $W \subseteq V$ is simply a closed subspace. We write $\mathrm{Pol}(V)$ for the set of polarizations in V. A *Lagrangian* in H is a closed subspace $L \subset H$ such that $H = L \oplus \alpha(L)$. We write $\mathrm{Lag}(H)$ for the set of Lagrangians in H. △

Given a polarization $W \in \mathrm{Pol}(V)$, we obtain a Lagrangian $L_W = W \oplus (W^\perp)^* \in \mathrm{Lag}(\widehat{V})$. The assignment $L_\bullet : \mathrm{Pol}(V) \to \mathrm{Lag}(\widehat{V})$ is injective, but far from surjective. Indeed, if $T : V \to V$ is a skew-adjoint, conjugate-linear map, then $\mathrm{graph}(\iota T)$ is a Lagrangian.

To see this, pick $v, w \in V$ arbitrary, and compute

$$\langle (v, \iota T v), (Tw, \iota w)\rangle = \langle v, Tw\rangle + \langle \iota T v, \iota w\rangle = \langle v, Tw\rangle + \langle w, Tv\rangle = \langle v, Tw\rangle - \langle v, Tw\rangle = 0.$$

This proves that $\mathrm{graph}(\iota T) \subseteq \alpha(\mathrm{graph}(\iota T))^\perp$. On the other hand, suppose that $(x, y) \in \alpha(\mathrm{graph}(\iota T))^\perp \subset V \oplus V^*$. We then have

$$0 = \langle (Tw, \iota w), (x, y)\rangle = \langle Tw, x\rangle + \langle \iota w, y\rangle = -\langle Tx, w\rangle + \langle \iota^* y, w\rangle = \langle \iota^* y - Tx, w\rangle.$$

Because this must hold for all $w \in V$, we have $\iota^* y - Tx = 0$, or in other words $y = \iota T x$, and thus $(x, y) = (x, \iota T x) \in \mathrm{graph}(\iota T)$.

If T is not the zero map, then $\mathrm{graph}(\iota T)$ is not in the image of $L_\bullet$.

Let $L \subset H$ be a Lagrangian. If $v \in L$, we write $v \wedge \bullet$ for the bounded operator $\Lambda L \to \Lambda L, f \mapsto v \wedge f$. If $w \in \alpha(L)$, we write $b(w)$ for the complex-linear extension of the map

$$\wedge^{n+1} L \to \wedge^n L : l_0 \wedge ... \wedge l_n \mapsto \sum_{k=0}^{n} (-1)^k \langle l_k, \alpha(w)\rangle l_0 \wedge ... \wedge \widehat{l_k} \wedge ... \wedge l_n$$

The map

$$\begin{aligned}\rho_L : H = L \oplus \alpha(L) &\to \mathcal{B}(\Lambda L),\\ (v, w) &\mapsto v \wedge \bullet + b(w)\end{aligned}$$

is a Clifford map, and thus extends to $*$-homomorphism $\rho_L : \mathrm{Cl}(H) \to \Lambda L$. The map ρ_L is the *Fock representation* of $\mathrm{Cl}(H)$ with respect to L.

Let $W \subseteq V$ be a polarization we then obtain a Lagrangian $L_W := W \oplus (W^\perp)^* \subset \widehat{V}$, and thus a corresponding Fock representation $\rho_{L_W} : \mathrm{Cl}(\widehat{V}) \to \mathcal{B}(\Lambda L_W)$. Pre-composition with the $*$-isomorphism u thus yields a representation $\rho_{L_W} u : \mathrm{CAR}(V) \to \mathcal{B}(\Lambda L_W)$.

A representation $\pi_W \, \mathrm{CAR}(V) \to \mathcal{B}(\Lambda L_W)$ can also be constructed directly as follows. First, observe that $\Lambda L_W = \Lambda W \otimes \Lambda(W^\perp)^*$. If $w \in W^\perp$, then we write $c(w)$ for the complex-linear extension of the map

$$\begin{aligned} \wedge^{n+1}(W^\perp)^* &\to \wedge^n(W^\perp)^*, \\ f_0 \wedge ... \wedge f_n &\mapsto \sum_{k=0}^{n} (-1)^k f_k(w) f_0 \wedge ... \wedge \widehat{f_k} \wedge ... \wedge f_n \end{aligned}$$

The map

$$\begin{aligned} \pi_W : V = W \oplus W^\perp &\to \mathcal{B}(\Lambda W \otimes \Lambda(W^\perp)^*), \\ (v, w) &\mapsto (v \wedge \bullet) \otimes \mathbb{1} + \mathbb{1} \otimes c(w) \end{aligned}$$

is then a CAR-map, whence it extends to a $*$-homomorphism $\pi_W : \mathrm{CAR}(V) \to \mathcal{B}(\Lambda L_W)$.

Proposition 3.2. $\pi_W = \rho_{L_W} u$

Proof. Let $(v, w) \in W \oplus W^\perp = V$ be arbitrary. We then have $j(v, w) = ((v, 0), (0, w)) \in (W \oplus (W^\perp)^*) \oplus (W^* \oplus W^\perp) = L_W \oplus \alpha(L_W) = \widehat{V}$. The map u is the (unique) extension of the map $i_{\widehat{V}} j : V \to \mathrm{Cl}(\widehat{V})$. We then compute

$$\rho_{L_W} j(v, w) = \rho_{L_W}((v, 0), (0, w)) = v \wedge \bullet + b(0, w)$$

We claim that $\mathbb{1} \otimes c(w) = b(0, w)$ for all $w \in W^\perp$. Indeed, let $y = x \otimes f_0 \wedge ... \wedge f_n \in \Lambda W \otimes \wedge^{n+1}(W^\perp)^*$ be arbitrary, we then have

$$\begin{aligned} b(0, w)(y) &= x \otimes \sum_{k=0}^{n} (-1)^k \langle f_k, \alpha(0, w) \rangle f_0 \wedge ... \wedge \widehat{f_k} \wedge ... \wedge f_n \\ &= x \otimes \sum_{k=0}^{n} (-1)^k \langle f_k, \iota(w) \rangle f_0 \wedge ... \wedge \widehat{f_k} \wedge ... \wedge f_n \\ &= x \otimes \sum_{k=0}^{n} (-1)^k f_k(w) f_0 \wedge ... \wedge \widehat{f_k} \wedge ... \wedge f_n \\ &= (\mathbb{1} \otimes c(w))(y) \end{aligned}$$

We thus see that $\pi_W(v, w) = \rho_{L_W} u(v, w)$ for all $(v, w) \in W \oplus W^\perp$. This implies that this identity must also hold on all of $\mathrm{CAR}(V)$. □

Remark 3.3. As before, we consider a complex Hilbert space H with real structure α. Let $L \in \mathrm{Lag}(H)$. Implicit in the definition of the Fock representation of $\mathrm{Cl}(H)$ on ΛL is the identification of $\alpha(L)$ with L^* through the (complex-linear) map $\alpha(L) \to L^*, w \mapsto \langle \bullet, \alpha(w) \rangle$. In fact, this identification allows us to identify H with $\widehat{L}$ (as complex Hilbert spaces with real structures). Proposition 2.7 then tells us that the inclusion $L \to H$ extends to an isomorphism of C*-algebras $u : \mathrm{CAR}(L) \to \mathrm{Cl}(H)$. We now define the $*$-isomorphism $s_L : \mathrm{CAR}(H) \to \mathrm{Cl}(H) \otimes \mathrm{Cl}(H^*)$ to be the composition of the following $*$-isomorphisms:

$$\mathrm{CAR}(H) = \mathrm{CAR}(L \oplus \alpha(L)) \to \mathrm{CAR}(L) \otimes \mathrm{CAR}(L^*) \to \mathrm{Cl}(H) \otimes \mathrm{Cl}(H^*).$$

It should be emphasized that the construction of s_L required the choice of a Lagrangian $L \in \mathrm{Lag}(H)$; moreover, different choices of Lagrangians lead to different isomorphisms. Let $\pi_L^L : \mathrm{CAR}(L) \to \mathcal{B}(\Lambda L)$ be the Fock representation of $\mathrm{CAR}(L)$ with respect to $L \in \mathrm{Pol}(L)$, and let π_L^H be the Fock representation of $\mathrm{CAR}(H)$ with respect to $L \in \mathrm{Pol}(H)$. As a corollary of Proposition 3.2 the map $u : \mathrm{CAR}(L) \to \mathrm{Cl}(H)$ satisfies $\pi_L^L = \rho_L u$, which in turn implies that $\pi_L^H = (\rho_L \otimes \rho_{L^*}) s_L$. △

References

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[Ten17] James E. Tener. "Construction of the unitary free fermion Segal CFT". *Commun. Math. Phys.*, 355(2):463–518, 2017. arXiv: 1608.02095. http://arxiv.org/abs/1608.02095, doi:10.1007/s00220-017-2959-x.