The spinor bundle on loop space & its fusion product

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Overview

1. Finite dimensional Clifford algebras and spinors

2. Clifford algebras and spinors in infinite dimensions

3. The spinor bundle on loop space

4. Fusion & Locality

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4. Fusion & Locality

Finite dimensional real Clifford algebras

V a finite dimensional Euclidean space

Definition: $\mathrm{Cl}(V)$

The *Clifford algebra* $\mathrm{Cl}(V)$ is the algebra generated by V subject to

$$vw + wv = 2\langle v, w\rangle \mathbf{1}, \quad v, w \in V.$$

$V \hookrightarrow \mathrm{Cl}(V)$

Lemma: Universal property

If A is a unital algebra, and $f : V \to A$ satisfies

$$f(v)f(w) + f(w)f(v) = 2\langle v, w\rangle \mathbf{1}, \quad v, w \in V,$$

then there is a unique homomorphism $F : \mathrm{Cl}(V) \to A$ extending f.

The spin group

Definition: $\mathrm{Spin}(d)$

The spin group $\mathrm{Spin}(d)$ is the universal cover of $\mathrm{SO}(d)$. I.e. $\mathrm{Spin}(d)$ is a simply connected Lie group which fits in the exact sequence

$$\mathbb{Z}_2 \hookrightarrow \mathrm{Spin}(d) \twoheadrightarrow \mathrm{SO}(d).$$

The spin group and the Clifford algebra

The spin group is a subgroup of the unit group of the Clifford algebra:

$$\mathrm{Spin}(d) \subset \mathrm{Cl}(\mathbb{R}^d)^\times.$$

Thus, $\mathrm{Spin}(d)$ acts on $\mathrm{Cl}(\mathbb{R}^d)$ by conjugation.

Where do spin groups & Clifford algebras appear?

- K-theory (Karoubi)
- Index theory (Atiyah-Singer index theorem)

  ⋮

- Quantum mechanics of particles with spin

Let $\mathcal{F}$ be a module for $\mathrm{Cl}(\mathbb{R}^d)$, and e_j the canonical basis for $\mathbb{R}^d$. Then spinor fields $\psi \in L^2(\mathbb{R}^d, \mathcal{F})$ satisfy the Dirac equation:

$$i \sum_{j=1}^{d} e_j \partial_j \psi = m\psi.$$

Spinors on Manifolds

Spinor bundles

Let M be a finite dimensional oriented Riemannian manifold. The Clifford algebras $\mathrm{Cl}(T_xM)$ fit together into a bundle $\mathrm{Cl}(TM) \to M$. If M is spin, then we can construct a bundle $\mathbb{S} \to M$ of modules for $\mathrm{Cl}(TM)$.

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$\vdots$

- Quantum mechanics of strings with spin

Build your own spinor bundle

M some finite dimensional, oriented "spacetime" manifold.

- Pick a representation $\mathcal{F}$ for $\mathrm{Cl}(\mathbb{R}^d)$ and thus also for $\mathrm{Spin}(d) \subset \mathrm{Cl}(\mathbb{R}^d)$.
- Reduce the structure group of $\mathrm{SO}(M)$ to $\mathrm{Spin}(d)$, call the result $\mathrm{Spin}(M)$.

Then $\mathbb{S} := \mathrm{Spin}(M) \times_{\mathrm{Spin}(d)} \mathcal{F}$ is a spinor bundle.

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Infinite dimensional Clifford algebras

V a complex Hilbert space, now *infinite* dimensional.

- Define a Clifford algebra as before.

Infinite dimensional Clifford algebras

V a complex Hilbert space, now *infinite* dimensional.

- Define a Clifford algebra as before.
- Complete this algebra to a C*-algebra.

This Clifford C*-algebra is universal.

Fock spaces

Definition: Lagrangians

A subspace $L \subset V$ is *Lagrangian* if

- $V = L \oplus \overline{L}$,
- $\langle v, \overline{w} \rangle = 0$, for all $v, w \in L$.

Identify $\overline{L} \simeq L^*$, through $w \mapsto \langle \overline{w}, \bullet \rangle$.

Definition: Fock representation

The *Fock space* $\mathcal{F}$ is the Hilbert completion of the exterior algebra ΛL.

- L acts on ΛL by left multiplication (creation).
- $\overline{L}$ acts on ΛL by contraction (annihilation).

This extends to a representation $\pi : \mathrm{Cl}(V) \to \mathcal{B}(\mathcal{F})$.

Bogoliubov transformations

$\mathrm{O}(V)$ is the orthogonal group. Let $g \in \mathrm{O}(V)$
Using the universal property, define the *Bogoliubov transformation* $\theta_g \in \mathrm{Aut}(\mathrm{Cl}(V))$ to be the extension of g.

Question: When are π and $\pi \circ \theta_g$ unitarily equivalent? I.e. when does there exist a $U \in \mathrm{U}(\mathcal{F})$ such that

$$U\pi(a)U^* = \pi(\theta_g a), \quad a \in \mathrm{Cl}(V).$$

In this case: call θ_g *implementable*.

Segal-Shale-Stinespring: Decompose g with respect to $V = L \oplus \overline{L}$, then θ_g is implementable if and only if its off-diagonal part $\overline{L} \to L$ is Hilbert-Schmidt.

A projective representation

Definition: Restricted orthogonal group

The *restricted orthogonal group* is

$$\mathrm{O}_{\text{res}}(V) := \{g \in \mathrm{O}(V) \mid g \text{ is implementable}\}.$$

If $g \in \mathrm{O}_{\text{res}}(V)$, and U implements g, then so does λU, for $\lambda \in \mathrm{U}(1)$. Hence, $\mathrm{O}_{\text{res}}(V)$ acts projectively in $\mathcal{F}$. This gives a central extension

$$\mathbf{1} \to \mathrm{U}(1) \to \mathrm{Imp}(V) \to \mathrm{O}_{\text{res}}(V) \to \mathbf{1}.$$

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Spin structures on loop spaces

Let M be a spin manifold with spin frame bundle $\mathrm{Spin}(M)$. Then $L\,\mathrm{Spin}(M)$ has structure group $L\,\mathrm{Spin}(d)$. The group $L\,\mathrm{Spin}(d)$ has a "basic" central extension:

$$\mathrm{U}(1) \to \widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d)$$

Definition (Killingback '87)

A *spin structure on the loop space* LM is a lift

$$\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M).$$

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A *spin structure on the loop space* LM is a lift

$$\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M).$$

Find a Hilbert space V with compatible $\widetilde{L\,\mathrm{Spin}}(d)$ and $\mathrm{Cl}(V)$ action.

The odd spinors on the circle

The Hilbert space

Set $V := L^2(S^1, \mathbb{C}^d)$. An (unorthodox) orthonormal basis of V is

$$\xi_{n,j} : e^{i\varphi} \mapsto e^{i(n+1/2)\varphi} e_j, \qquad \varphi \in [0, 2\pi]$$

where $n \in \mathbb{Z}$ and $\{e_j\}_{j=1,\dots,d}$ the standard basis of $\mathbb{C}^d$.
Pointwise complex conjugation gives: $\overline{\xi_{n,j}} = \xi_{-n-1,j}$.

Let $\mathbb{S} \to S^1$ be the odd spinor bundle on the circle. The basis $\xi_{n,j}$ looks more natural when V is identified with $L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d)$.

A Lagrangian

Compute for $n, m \geqslant 0$ and $j, l = 1, ..., d$,

$$\begin{aligned} \langle \xi_{n,j}, \overline{\xi_{m,l}} \rangle &= \langle \xi_{n,j}, \xi_{-m-1,l} \rangle \\ &= \delta_{n,-m-1}\, \delta_{j,l} && \xi\text{'s are orthonormal} \\ &= 0. && n \neq -m-1 \end{aligned}$$

Definition: Atiyah-Patodi-Singer (APS) Lagrangian

The *APS Lagrangian* $L \subset V$ is the closure of the span of the vectors $\xi_{n,j}$ with $n \geqslant 0$ and $j = 1, ..., d$.

Have a Clifford algebra $\mathrm{Cl}(V)$, and a Fock space $\mathcal{F}$ corresponding to the Lagrangian L.

The basic central extension of $L\,\mathrm{Spin}(d)$

The action of $L\,\mathrm{SO}(d)$

The loop group $L\,\mathrm{SO}(d)$ acts on $V = L^2(S^1, \mathbb{C}^d)$ pointwise.
Pressley-Segal: $L\,\mathrm{SO}(d) \to \mathrm{O}_{\mathrm{res}}(V)$.

Lemma

The pullback

$$\begin{array}{ccccc}
\widetilde{L\,\mathrm{Spin}(d)} & \longrightarrow & & \longrightarrow & \mathrm{Imp}(V) \\
\downarrow & & & & \downarrow \\
L\,\mathrm{Spin}(d) & \longrightarrow & L\,\mathrm{SO}(d) & \longrightarrow & \mathrm{O}_{\mathrm{res}}(V)
\end{array}$$

is the basic central extension of $L\,\mathrm{Spin}(d)$

The spinor bundle on loop space

Have representations $\widetilde{L\,\mathrm{Spin}}(d) \circlearrowright \mathcal{F}$ and $L\,\mathrm{Spin}(d) \circlearrowright \mathrm{Cl}(V)$.

Define the *Clifford bundle on loop space*:
$L\,\mathrm{Spin}(M) \times_{L\,\mathrm{Spin}(d)} \mathrm{Cl}(V) =: \mathrm{Cl}(LM) \to LM$.
Each fibre is a Clifford algebra.

Given a "spin structure on loop space": $\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$, we

Define the *Spinor bundle on loop space*:
$\widetilde{L\,\mathrm{Spin}}(M) \times_{\widetilde{L\,\mathrm{Spin}}(d)} \mathcal{F} =: \mathcal{F}(LM) \to LM$. This is a bundle of rigged Hilbert spaces, where each fibre is a Fock space.

The Clifford bundle acts on the Fock bundle fibrewise.

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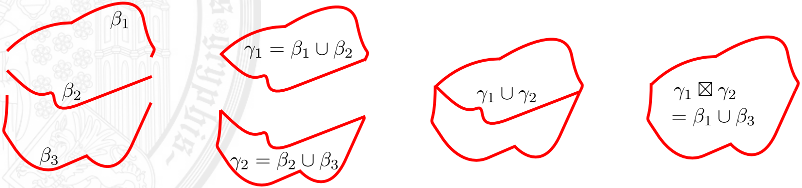
Smooth loop space of a manifold

Definition

The space of *smooth loops/paths*:

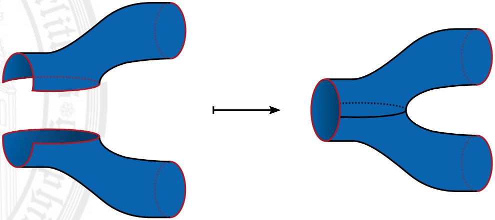
$$LM = C^\infty(S^1, M) \qquad PM = C^\infty([0,\pi], M)$$



Loops/paths that can be glued together are called *compatible*.

Interacting strings

Strings interact through splitting and merging.
Want to relate the fibres $\mathcal{F}(LM)_{\gamma_1}$ and $\mathcal{F}(LM)_{\gamma_2}$ with the fibre $\mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2}$.



Will need more structure for this to work. From now on: M is equipped with a string structure. A string structure on M transgresses to a "fusive" spin structure on LM.

Clifford bundles over path space

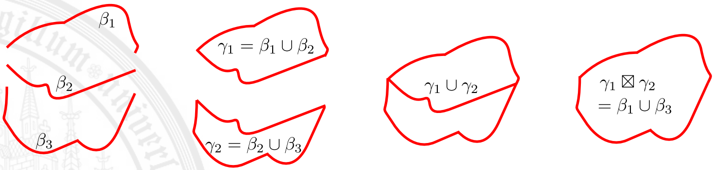
Split S^1 into the upper/lower semi-circle, $I_\pm$.

- $V_\pm := \{f \in V \mid \mathrm{supp}(f) \subseteq I_\pm\}$.
- $P\,\mathrm{Spin}(d) \circlearrowright \mathrm{Cl}(V_\pm)$.
- $P\,\mathrm{Spin}(M) \times_{P\,\mathrm{Spin}(d)} \mathrm{Cl}(V_\pm) =: \mathrm{Cl}_\pm(PM) \to PM$.

Fix $\beta_1 \cup \beta_2 = \gamma \in LM$, for $\beta_1, \beta_2 \in PM$, then $\mathcal{F}(LM)_\gamma$ is a left $\mathrm{Cl}(LM)_\gamma = \mathrm{Cl}_+(PM)_{\beta_1} \otimes \mathrm{Cl}_-(PM)_{\beta_2}$-module.
We turn it into a bimodule:

$$\mathrm{Cl}_+(PM)_{\beta_1} \circlearrowright \mathcal{F}(LM)_\gamma \circlearrowleft \mathrm{Cl}_+(PM)_{\beta_2}.$$

Fusion



Assume M is string.

Theorem: Fusion of Fock spaces [PK'20]

For each triple $(\beta_1, \beta_2, \beta_3)$, there exists an isomorphism of $\mathrm{Cl}_+(PM)_{\beta_1}$-$\mathrm{Cl}_+(PM)_{\beta_3}$ bimodules

$$\mu_{1,2,3} : \mathcal{F}(LM)_{\gamma_1} \boxtimes_{\mathrm{Cl}_+(PM)_{\beta_2}} \mathcal{F}(LM)_{\gamma_2} \xrightarrow{\simeq} \mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2}.$$

such that these isomorphisms are associative, i.e. have a commutative square for each quadruple $(\beta_1, \beta_2, \beta_3, \beta_4)$.

Sketch of construction

- Find a natural isomorphism of the canonical fibre: $\mathcal{F} \boxtimes_{\mathrm{Cl}_+(V)} \mathcal{F} \xrightarrow{\simeq} \mathcal{F}$.
- Choose suitable $\varphi_1, \varphi_2, \varphi_3$

$$
\begin{array}{ccc}
\mathcal{F}(LM)_{\gamma_1} \boxtimes_{\mathrm{Cl}_+(PM)_{\beta_2}} \mathcal{F}(LM)_{\gamma_2} & \overset{\mu_{\varphi_i}}{\dashrightarrow} & \mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2} \\
\downarrow \varphi_1 \boxtimes \varphi_2 & & \uparrow \varphi_3^{-1} \\
\mathcal{F} \boxtimes_{\mathrm{Cl}_+(V)} \mathcal{F} & \longrightarrow & \mathcal{F}
\end{array}
$$

Picking φ_i arbitrarily will fail associativity.
Solution: for any φ_1, φ_2 produce φ_3.

Finding suitable φ_i

An element $p \in \widetilde{L\,\mathrm{Spin}(M)}_\gamma$ gives a trivialization:

$$\varphi_p : \mathcal{F}(LM)_\gamma \to \mathcal{F}, [p, v] \mapsto v.$$

- Pick compatible loops $\alpha_1 \in L\,\mathrm{Spin}(M)_{\gamma_1}$ and $\alpha_2 \in L\,\mathrm{Spin}(M)_{\gamma_2}$.
- Set $\alpha_3 = \alpha_1 \boxtimes \alpha_2 \in L\,\mathrm{Spin}(M)_{\gamma_3}$.
- Waldorf: Because $\widetilde{L\,\mathrm{Spin}(M)} \to L\,\mathrm{Spin}(M)$ comes from a *string structure* on M, it is a *fusion* extension: $\Rightarrow$ given $p_1, p_2 \in \widetilde{L\,\mathrm{Spin}(M)}$ with $p_1 \mapsto \alpha_1$ and $p_2 \mapsto \alpha_2$, can construct $p_3 \in \widetilde{L\,\mathrm{Spin}(M)}$ with $p_3 \mapsto \alpha_3$.

Claim: The map μ_{φ_i} does not depend on any of the choices.

Summary

- Studied infinite dimensional Clifford algebras.
- Constructed the spinor bundle on loop space $\mathcal{F}(LM) \to LM$.
- Constructed an associative family of fusion isomorphisms $\mathcal{F}(LM)_{\gamma_1} \boxtimes_{\mathrm{Cl}_+(PM)_{\beta_2}} \mathcal{F}(LM)_{\gamma_2} \xrightarrow{\simeq} \mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2}$ expressing that $\mathcal{F} \to LM$ is *local* in M.

Further work

- "Regress" the bundle $\mathcal{F} \to LM$ to a (2-vector) "stringor" bundle over M.
- The diffeomorphism group of the circle acts in LM. Lift this action to a bundle action $\mathcal{F} \to LM$.
- Equip $\mathcal{F} \to LM$ with a notion of parallel transport over surfaces.

The Dirac operator

- Set $\gamma_i = e_i$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^d$.
- Fix a representation $\mathcal{F}$ of $\mathrm{Cl}(\mathbb{R}^d)$.
- Replace $C^\infty(\mathbb{R}^d)$ by $C^\infty(\mathbb{R}^d, \mathcal{F})$.

$$D := \sum_{j=1}^{d} \gamma_j \left(\frac{\partial}{\partial x_j} \right) \quad \Longrightarrow \quad D^2 = \Delta = \sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} \right)^2 .$$

D is the *Dirac operator*.

Action of $\mathrm{Cl}(V)$ on $\mathcal{F}$

Let $v + w \in L \oplus \alpha(L) = V \subset \mathrm{Cl}(V)$ and $l_1 \wedge \cdots \wedge l_k \in \mathcal{F}$ then:

$$\pi(v+w)(l_1 \wedge \ldots \wedge l_k) = v \wedge l_1 \wedge \cdots \wedge l_k + \sum_{j=1}^{k} (-1)^{j-1} \langle l_j, \alpha(w) \rangle l_1 \wedge \cdots \wedge \hat{l}_j \wedge \cdots \wedge l_k$$

The Dirac operator on a spin manifold

M a spin manifold with spinor bundle $\mathbb{S} \to M$. Let ∇ be the lift of the Levi-Civita connection to $\mathbb{S}$.

- View ∇ as a map $\nabla : \Gamma(\mathbb{S}) \to \Gamma(T^*M) \otimes \Gamma(\mathbb{S})$.
- Identify T^*M with TM and embed TM into $\mathrm{Cl}(M)$.
- Write $c : \Gamma(\mathrm{Cl}(M)) \otimes \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ for Clifford multiplication.
- The Dirac operator is the map

$$D : \Gamma(\mathbb{S}) \xrightarrow{\nabla} \Gamma(\mathrm{Cl}(M)) \otimes \Gamma(\mathbb{S}) \xrightarrow{c} \Gamma(\mathbb{S}).$$

In local coordinates

$$Ds = \sum_i e_i \cdot \nabla_{e_i} s,$$

where e_i forms a basis for TM.

Rigged Hilbert space

Definition

A *rigged Hilbert space* is a triple (F, H, ι) consisting of

- A Hilbert space H.
- A Fréchet space F.
- A continuous linear injection $\iota : F \to H$.

Such that the image $\iota(F)$ is dense in H.

The Atiyah-Bott-Shapiro isomorphism

A_n the Grothendieck group of real graded $\mathrm{Cl}(\mathbb{R}^n)$ modules. In fact A_n is a ring.

Theorem: Atiyah-Bott-Shapiro ('63)

For each k there is an isomorphism:

$$A_n / i^* A_{n+1} \xrightarrow{\simeq} KO^{-k}(\mathrm{pt}).$$

The Atiyah-Singer index theorem

If P is an elliptic operator between two vector bundles E and F on a compact manifold, then the topological index of P is equal to the analytical index of P.