Cutting cakes

Or: How to share your cake and eat it too.

‘Tis the season

Heterogeneous cakes

One piece may have different relative values to different people.

I cut, you choose

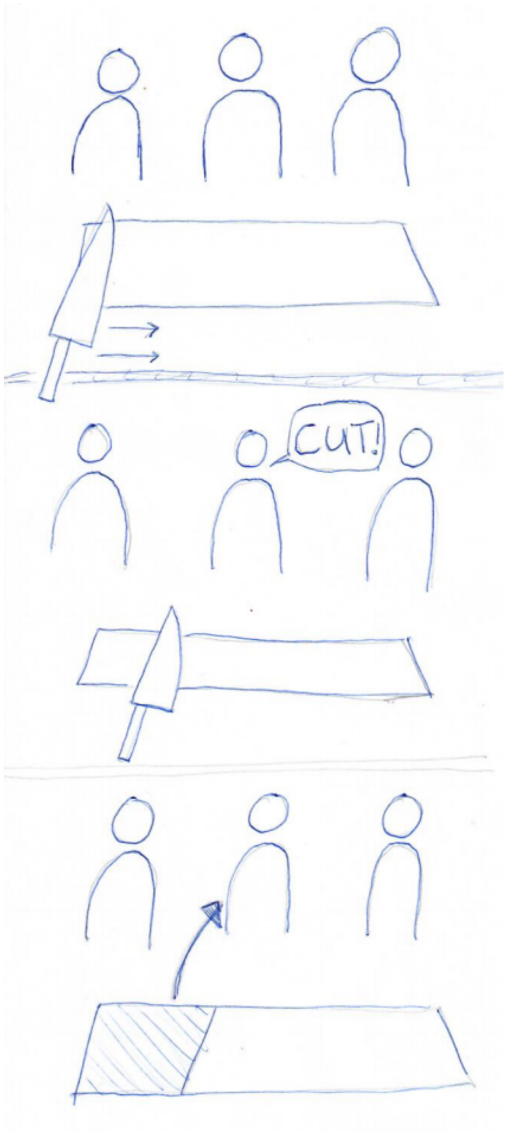
Two people share a cake.

One person cuts the cake in half, the other person chooses.

Everyone is happy.

Dubins and Spanier's moving knife*

- Slowly move a knife over the cake.
- One person yells "cut!".
- This person gets the slice just cut off.
- Repeat until there is just one person left.



Works for any number, *n*, of players!

The result is a *proportional* distribution of the cake:
Each person has at least *1/n* of the cake.

The result might not be *envy-free*:
Someone might prefer someone else's piece.

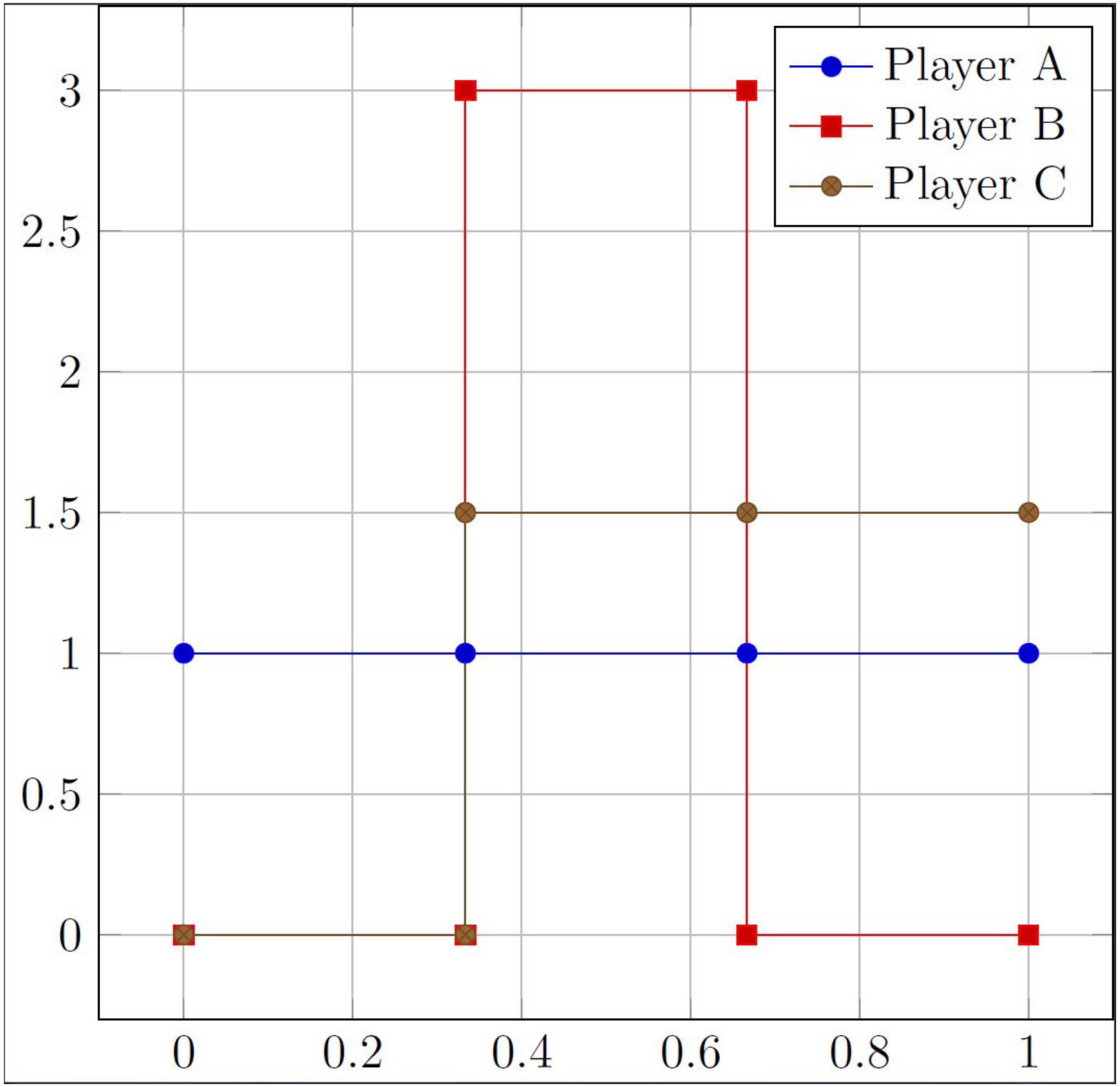
*L. Dubins and E. Spanier, How to cut a cake fairly, *Amer. Math. Montly* **68** (1961) 1-17

Envious moving knife

- Player A yells cut at *1/3*
- Player B yells cut at *1/2*

  The distribution is *proportional*.

  It is not *envy-free*.



Solution: Give everyone a knife!

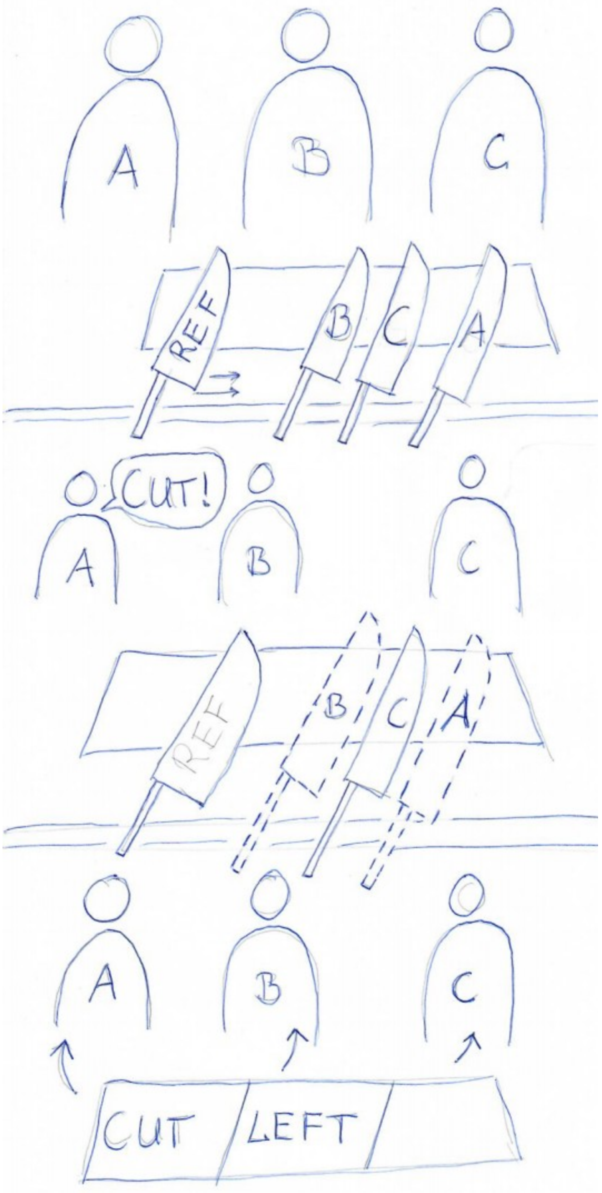
Stromquist's moving knives*

- Referee slowly moves a knife.
- Hungry people A,B,C divide right side of the cake.
- Someone yells "Cut!".
- The yeller gets the left piece.
- Person with their knife closest to the referee gets the middle piece.

Works only for three people.

Results in proportional and envy-free solution.

Could be a bit dangerous, don't try this at your niece's sixth birthday party!



*W. Stromquist, How to Cut a Cake Fairly. *Amer. Math. Monthly.* **87** (8): 640 (1980)

Framework

- The cake is some set X.
- Each player has a normalized measure on X, representing how he values each possible piece of cake.
- The value of two disjoint pieces of cake is the sum of the values of the pieces.
- The measure is non-negative. It's never bad to have more cake!
- The measure is atomless, no single point has non-zero value.

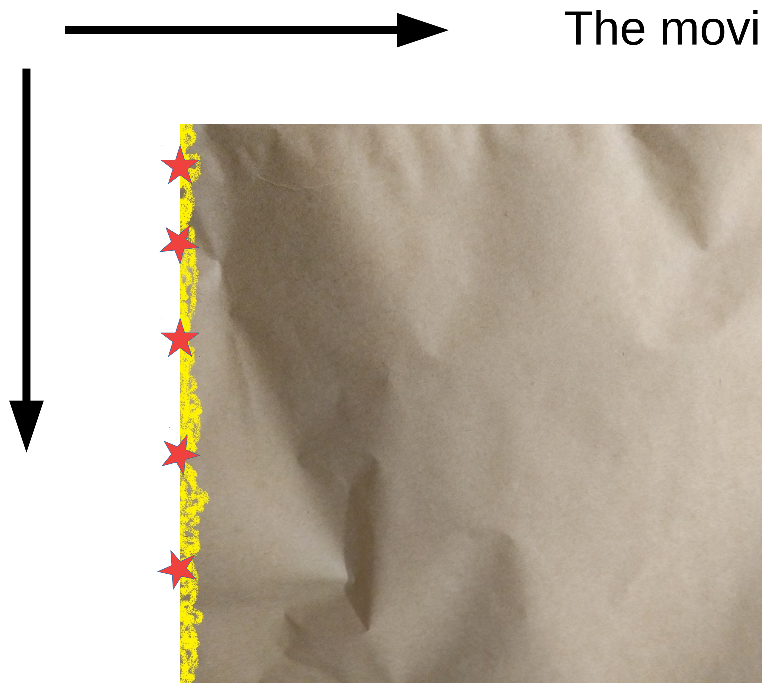
Hidden assumption: absolute continuity.

Imagine a cake

bland as cardboard, but with

delicious frosting.

The moving knife procedure fails.



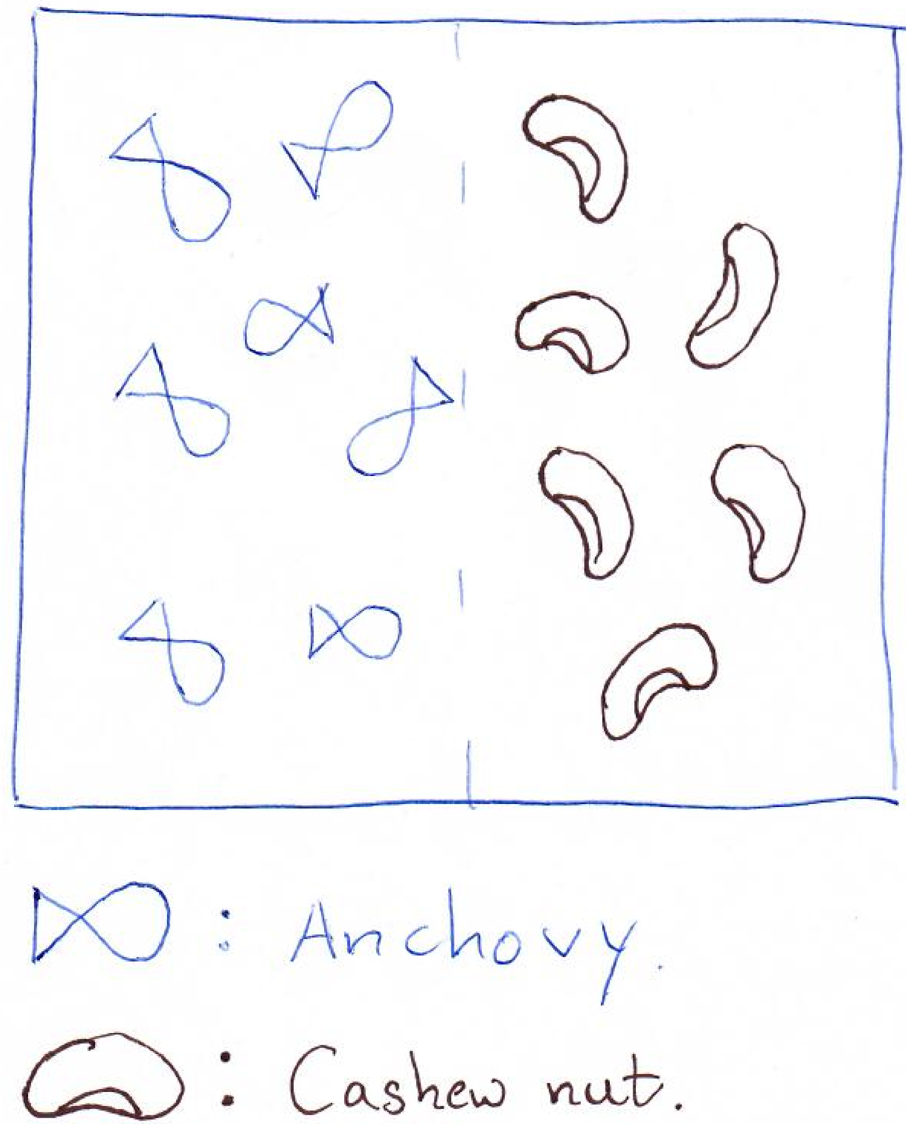
Better than fair.

Imagine a cake, one half with anchovies and one half with cashew nuts (and try not to be sick).

Suppose that I like cashew nuts, but I'm allergic to seafood. And you like anchovies, but you're allergic to nuts.

The "I cut, you choose" protocol might give us a fair division, but it's hardly the best one.

- A division is *weak Pareto optimal* if there is no division that is better for everyone.
- A division is *strong Pareto optimal* if there is no division that is better for at least one person and no worse for the rest.



Surplus Procedure

Goal: If possible, give both players more than 50%.

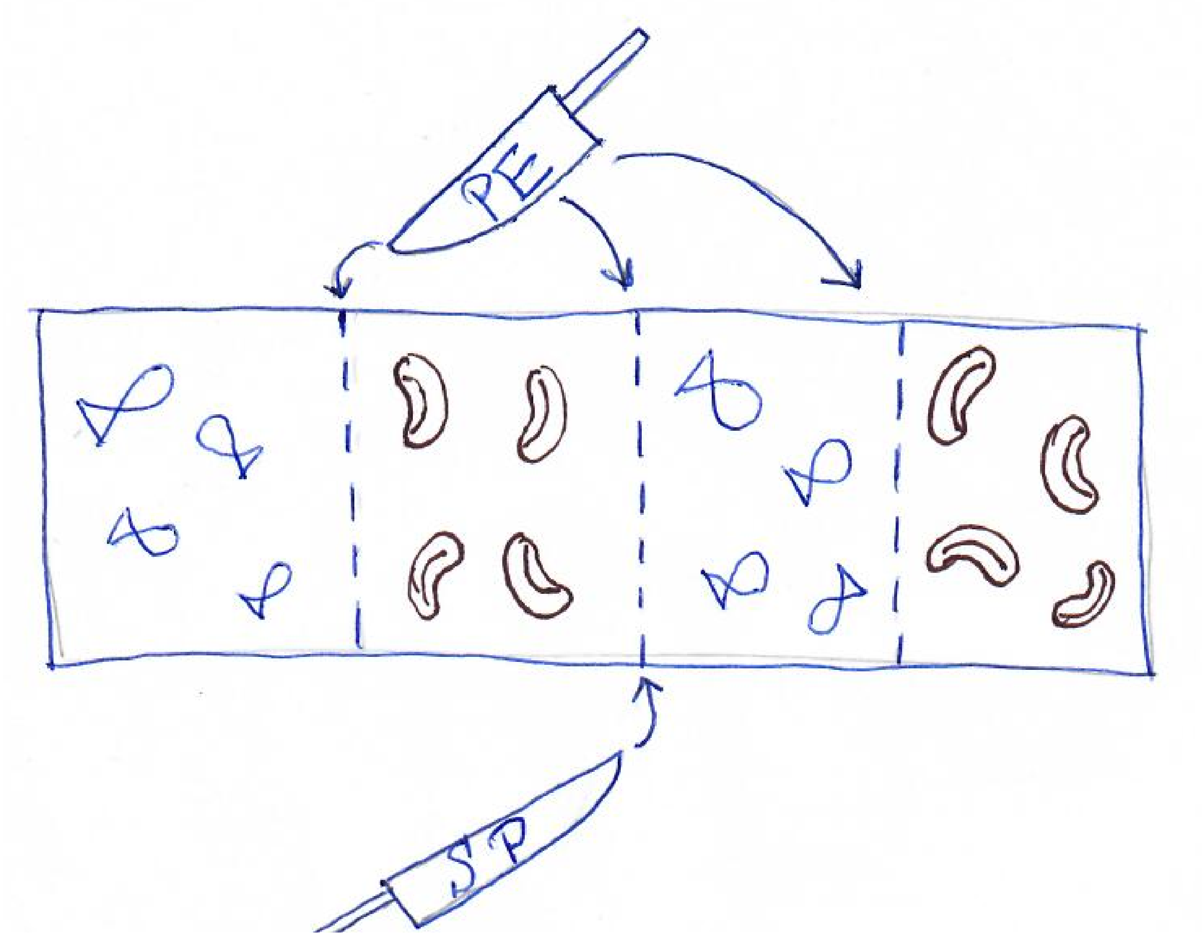
- Both players tell their measure to a referee.
- The referee determines the medians: a and b.
- The referee cuts the cake at a specific point c in between a and b.

Challenge: How does one decide c? One option:

Solution should be
- Fair
- Envy-free
- *Relatively equitable*

Disconnected pieces?

If we allow more than two pieces the surplus procedure is not optimal!



Liars take the cake.

The surplus procedure is not *strategy-proof*.

Lying can give a player a *risk-free* advantage.

Recall the cut point formula:

$$\frac{v_1(a,c)}{v_1(a,b)} = \frac{v_2(c,b)}{v_2(a,b)}$$

More players?

Stromquist's moving knifes procedure does not generalize to more than three players.

Envy-free procedure for n players *does* exist.*

This solution has a big flaw, it has *unbounded runtime*.

In fact: **Theorem** (Stromquist**)

-There is no bounded algorithm that gives an envy-free connected division of a one-dimensional cake for three or more players.

*F. E. Su, Rental Harmony: Sperner's lemma in fair division, *Amer. Math. Monthly* **106** (1999) 930-942

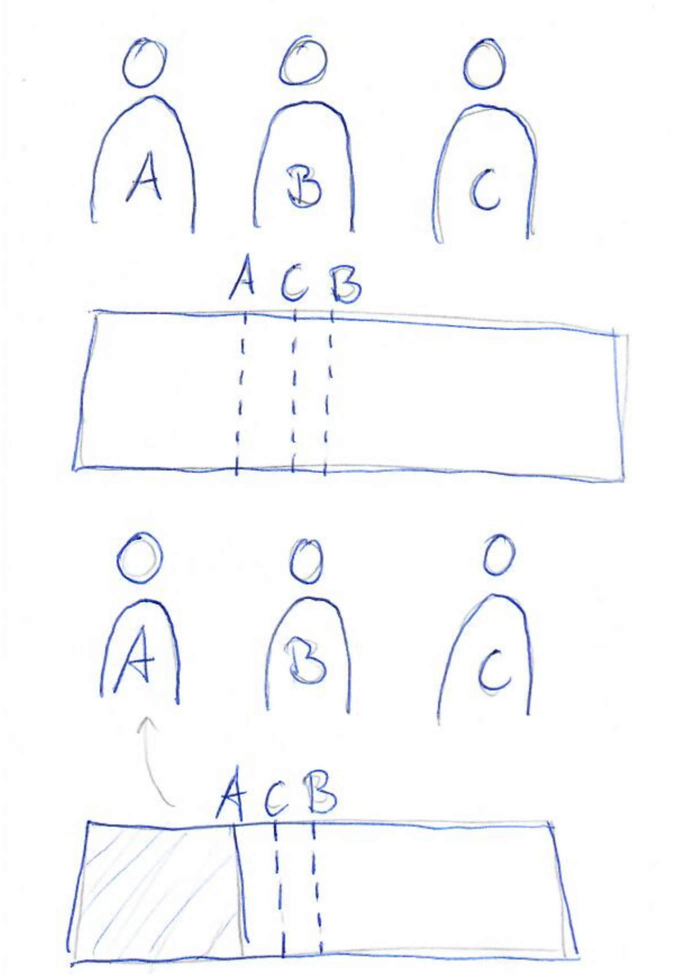
W. Stromquist, Envy-free cake divisions cannot be found by finite protocols, *Elec. J. of Comb.* **15

Bounded proportional division

Dubins and Spanier's moving knife is not finite either, but has a finite analog.

- Each player makes a mark at *1/n*
- The player with the left-most mark gets that piece
- Repeat.

(Complexity n^2)



Disconnected pieces?

- *n=3*: Selfridge-Conway* finds an envy-free distribution in a finite number of steps with at most five cuts.
- Arbitrary *n*: Aziz and Mackenzie**

$$n^{n^{n^{n^{n^{n}}}}}$$

*J. Robertson, W. Webb, Cake-Cutting Algorithms: Be Fair If You Can (1998)

**H. Aziz, S. Mackenzie, A Discrete and Bounded Envy-free Cake Cutting Protocol for Any Number of Agents, *Proc. of the 48th Annual ACM SIGACT Sym. on Theory of Computing* - STOC (2016) p. 454

Application: Politics

"Compromise is the art of dividing a cake in such a way that everyone believes he has the biggest piece."

- Ludwig Erhard