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Spinors on loop space

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Overview

1. Reductions and lifts of the frame bundle
2. Clifford algebras and Fock spaces
3. Representing $L\widetilde{\mathrm{Spin}}(d)$ in Fock space
4. Fusion of Fock spaces
5. Locality

The Whitehead tower of $\mathrm{O}(d)$

Homotopy groups of $\mathrm{O}(d)$ for $d = 3$ and $d \geqslant 5$

k	0	1	2	3	...
$\pi_k(\mathrm{O}(d))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	...

The Whitehead tower

$$\mathrm{O}(d) \leftarrow \mathrm{SO}(d) \leftarrow \mathrm{Spin}(d) \leftarrow \dots$$

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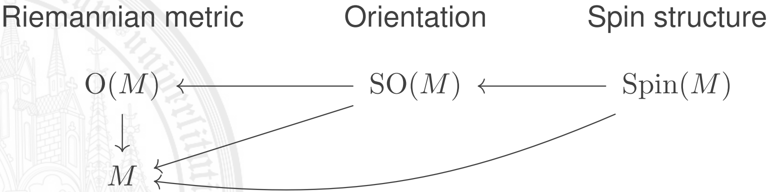
The Whitehead tower

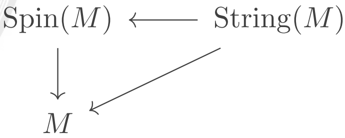
$$\mathrm{O}(d) \leftarrow \mathrm{SO}(d) \leftarrow \mathrm{Spin}(d) \leftarrow \mathrm{String}(d) \leftarrow \dots$$

$\mathrm{String}(d)$ is not a finite dimensional Lie group.
There are models: Fréchet Lie group, smooth 2-group...

Reduction and lifts



$G(M)$ is a principal G-bundle. Horizontal arrows are equivariant.
Morally, a string structure is a lift:



Characteristic classes

	SO	Spin	String
Obstruction	$w_1 \in H^1(M, \mathbb{Z}_2)$	$w_2 \in H^2(M, \mathbb{Z}_2)$	$p_1/2 \in H^4(M, \mathbb{Z})$
Enumeration	$H^0(M, \mathbb{Z}_2)$	$H^1(M, \mathbb{Z}_2)$	$H^3(M, \mathbb{Z})$

Transgression to loop space

$$\begin{array}{ccccc} H^4(M,\mathbb{Z}) & \xrightarrow{\mathrm{ev}^*} & H^4(S^1\times LM,\mathbb{Z}) & \xrightarrow{\int \mathrm{d}\,t} & H^3(LM,\mathbb{Z}), \\ p_1/2 & & \longmapsto & & \lambda \end{array}$$

Killingback: λ obstructs

$$\begin{array}{ccc} L\,\mathrm{Spin}(M) & \longleftarrow & \widetilde{L\,\mathrm{Spin}(M)} \\ \downarrow & \swarrow & \\ LM & & \end{array}$$

$$\mathbf{1}\to \mathrm{U}(1)\to \widetilde{L\,\mathrm{Spin}}(d)\to L\,\mathrm{Spin}(d)\to \mathbf{1}.$$

Vector bundles

Associated vector bundles

G	$\mathrm{O}(d)$	$\mathrm{SO}(d)$	$\mathrm{Spin}(d)$	$\widetilde{L\,\mathrm{Spin}}(d)$	$\mathrm{String}(d)$
V	$\mathbb{R}^d$	$\mathbb{R}^d$	Δ_d	??	??
$G(M) \times_G V$	TM	TM	$\mathbb{S}(M)$	??	??

What we expect $\mathrm{String}(M) \times_{\mathrm{String(d)}} V$ to be depends on the model of $\mathrm{String}(d)$.
Infinite dimensional vector bundle, 2-vector bundle, ...

Vector bundles

Associated vector bundles

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$G(M) \times_G V$	TM	TM	$\mathbb{S}(M)$	??	??

What we expect $\mathrm{String}(M) \times_{\mathrm{String(d)}} V$ to be depends on the model of $\mathrm{String}(d)$.
Infinite dimensional vector bundle, 2-vector bundle, ...

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Infinite dimensional Clifford algebras

V a complex Hilbert space, and $\alpha : V \to V$ a *real structure*, i.e. an isometry satisfying

$$\alpha^2 = \mathbf{1}, \text{ and } \quad \alpha(\lambda v) = \overline{\lambda}\alpha(v), \quad \lambda \in \mathbb{C}, v \in V.$$

If A is a C* algebra, then $f : V \to A$ is a *Clifford* map if

$$f(v)^* = f(\alpha(v)), \text{ and } f(v)f(w) + f(w)f(v) = \langle v, \alpha(w)\rangle \mathbf{1}, \quad v, w \in V.$$

Definition: Clifford C* algebra

The *Clifford C* algebra* $\mathrm{Cl}(V)$ is the universal C* algebra through which any Clifford map factors. I.e. if $f : V \to A$ is a Clifford map, then there exists a unique algebra homomorphism $F : \mathrm{Cl}(V) \to A$ extending f.

Fock spaces

Definition: Lagrangians

A subspace $L \subset V$ is *Lagrangian* if

- $V = L \oplus \alpha(L)$,
- $\langle v, \alpha(w) \rangle = 0$, for all $v, w \in L$.

Identify $\alpha(L) \simeq L^*$, through $w \mapsto \langle \alpha(w), \bullet \rangle$.

Definition: Fock representation

The *Fock space* $\mathcal{F}$ is the Hilbert completion of the exterior algebra ΛL. The map $\pi : L \oplus \alpha(L) \to \mathcal{B}(\mathcal{F}), (v, w) \mapsto v \wedge \bullet + \iota_w$ is a Clifford map. Its extension $\pi : \mathrm{Cl}(V) \to \mathcal{B}(\mathcal{F})$ is the *Fock representation*.

Bogoliubov transformations & 2^{nd} quantization

Set $\mathrm{O}(V) := \{g \in \mathrm{U}(V) \mid g\alpha = \alpha g\}$.
If $g \in \mathrm{O}(V)$, then $V \xrightarrow{g} V \to \mathrm{Cl}(V)$ is a Clifford map. Write $\theta_g \in \mathrm{Aut}(\mathrm{Cl}(V))$ for its extension.

Question: When is θ_g *implementable*? I.e. when does there exist a $U \in \mathrm{U}(\mathcal{F})$ such that

$$U\pi(a)U^* = \pi(\theta_g a), \quad a \in \mathrm{Cl}(V).$$

Segal-Shale-Stinespring: Decompose g with respect to $V = L \oplus \alpha(L)$:

$$g = \begin{pmatrix} g_1 & g_2 \\ \alpha g_2 \alpha & \alpha g_1 \alpha \end{pmatrix},$$

then θ_g is implementable if and only if $g_2 : \alpha(L) \to L$ is Hilbert-Schmidt.

A projective representation

Definition: Restricted orthogonal group

The *restricted orthogonal group* is

$$\mathrm{O}_{\text{res}}(V) := \{g \in \mathrm{O}(V) \mid g \text{ is implementable}\}.$$

If $g \in \mathrm{O}_{\text{res}}(V)$, and U implements g, then so does λU, for $\lambda \in \mathrm{U}(1)$. Hence, $\mathrm{O}_{\text{res}}(V)$ acts projectively in $\mathcal{F}$. This gives a central extension

$$\mathbf{1} \to \mathrm{U}(1) \to \mathrm{Imp}(V) \to \mathrm{O}_{\text{res}}(V) \to \mathbf{1}.$$

The odd spinors on the circle

The Hilbert space

Set $V := L^2(S^1, \mathbb{C}^d)$. An (unorthodox) orthonormal basis of V is

$$\xi_{n,j} : e^{i\varphi} \mapsto e^{i(n+1/2)\varphi} e_j, \qquad \varphi \in [0, 2\pi]$$

where $n \in \mathbb{Z}$ and $\{e_j\}_{j=1,\ldots,d}$ the standard basis of $\mathbb{C}^d$.
Define the real structure α as point-wise complex conjugation. Have $\alpha(\xi_{n,j}) = \xi_{-n-1,j}$.

Let $\mathbb{S} \to S^1$ be the odd spinor bundle on the circle. The basis $\xi_{n,j}$ looks more natural when V is identified with $L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d)$.

A Lagrangian

Compute for $n, m \geqslant 0$ and $j, l = 1, ..., d$,

$$\begin{aligned} \langle \xi_{n,j}, \alpha(\xi_{m,l}) \rangle &= \langle \xi_{n,j}, \xi_{-m-1,l} \rangle \\ &= \delta_{n,-m-1}\, \delta_{j,l} && \xi\text{'s are orthonormal} \\ &= 0. && n \neq -m-1 \end{aligned}$$

Definition: Atiyah-Patodi-Singer (APS) Lagrangian

The *APS Lagrangian* $L \subset V$ is the closure of the span of the vectors $\xi_{n,j}$ with $n \geqslant 0$ and $j = 1, ..., d$.

Have a Clifford algebra $\mathrm{Cl}(V)$, and a Fock space $\Lambda L^{\langle \cdot, \cdot \rangle}$.

The basic central extension of $L\,\mathrm{Spin}(d)$

The action of $L\,\mathrm{SO}(d)$

The loop group $L\,\mathrm{SO}(d)$ acts on $V = L^2(S^1, \mathbb{C}^d)$ pointwise.
Pressley-Segal: $L\,\mathrm{SO}(d) \to \mathrm{O}_{\mathrm{res}}(V)$.

Lemma

The pullback

$$\begin{array}{ccccc} \widetilde{L\,\mathrm{Spin}}(d) & & \longrightarrow & & \mathrm{Imp}(V) \\ \downarrow & & & & \downarrow \\ L\,\mathrm{Spin}(d) & \longrightarrow & L\,\mathrm{SO}(d) & \longrightarrow & \mathrm{O}_{\mathrm{res}}(V) \end{array}$$

is the basic central extension of $L\,\mathrm{Spin}(d)$

Associated bundles

Have representations $\widetilde{L\,\mathrm{Spin}}(d) \circlearrowright \mathcal{F}$ and $L\,\mathrm{Spin}(d) \circlearrowright \mathrm{Cl}(V)$.

Clifford bundle

Define the Clifford bundle $L\,\mathrm{Spin}(M) \times_{L\,\mathrm{Spin}(d)} \mathrm{Cl}(V) =: \mathrm{Cl}(LM) \to LM$. Each fibre is a Clifford algebra.

Fock bundle

Define the Fock bundle $\widetilde{L\,\mathrm{Spin}}(M) \times_{\widetilde{L\,\mathrm{Spin}}(d)} \mathcal{F} =: \mathcal{F}(LM) \to LM$. Each fibre is a Fock space.

The Clifford bundle acts on the Fock bundle fibrewise.

Fock space as a bimodule

Set $V_\pm := \{f \in V \mid \mathrm{supp}(f) \subseteq I_\pm\}$, then $\mathrm{Cl}(V) = \mathrm{Cl}(V_-) \otimes \mathrm{Cl}(V_+)$. And $\mathcal{F}$ becomes a graded $\mathrm{Cl}(V_-)$-$\mathrm{Cl}(V_+)^{\mathrm{op}}$ bimodule.

Want to consider $\mathcal{F} \otimes_{\mathrm{Cl}(V_\pm)} \mathcal{F}$, as a $\mathrm{Cl}(V_-)$-$\mathrm{Cl}(V_+)^{\mathrm{op}}$ bimodule. There are two problems:

- Need an isomorphism $\mathrm{Cl}(V_-) \to \mathrm{Cl}(V_+)^{\mathrm{op}}$.
- The algebraic tensor product $\mathcal{F} \otimes_{\mathrm{Cl}(V_\pm)} \mathcal{F}$ will in general not be a Hilbert space.

Solution: Work with tensor product of bimodules for *von Neumann algebras* instead of algebras.

Von Neumann algebras

Definition

If $A \subseteq \mathcal{B}(H)$ is a subalgebra of the bounded operators on H then the *commutant* of A is defined to be

$$A' := \{x \in \mathcal{B}(H) \mid [x, a] = 0 \text{ for all } a \in A\}.$$

Definition

A *von Neumann algebra* A is a subalgebra of $\mathcal{B}(H)$ for some Hilbert space H with the property that $A'' = A$.

If $A \subseteq \mathcal{B}(H)$, then $A'' = (A'')''$ is a von Neumann algebra and $A \subseteq A''$. A module for a von Neumann algebra is a Hilbert space in which the von Neumann algebra acts.

The standard form of a von Neumann algebra

Let $B \subseteq \mathcal{B}(H)$ be a von Neumann algebra. Let $\Omega \in H$ be a vector in H with the property that $B\Omega$ is dense in H and that $b\Omega = 0 \Rightarrow b = 0$ for all $b \in B$.

Definition

The *standard form* $L^2_\Omega(B)$ is defined to be the completion of B with respect to the inner product

$$B \times B \to \mathbb{C}, \quad (b, b') \mapsto (b\Omega, b'\Omega).$$

Multiplication on B extends to a left action: $B \times L^2_\Omega(B) \to L^2_\Omega(B)$.

Using Tomita-Takesaki theory, the Hilbert space $L^2_\Omega(B)$ is in a canonical way a B-B bimodule.

Connes fusion

- A, B, C von Neumann algebras
- H an A-B bimodule
- K a B-C bimodule

Then $H \boxtimes_B K$ is an A-C bimodule, called the Connes fusion of H with K.
The standard form of B is the unit for Connes fusion, i.e.

$$L^2_\Omega(B) \boxtimes_B K \cong K.$$

Fusion of Fock spaces

We define a conjugate linear involution τ of V by

$$(\tau f)(z) = \overline{f(\overline{z})}, \quad f \in V, z \in S^1.$$

The map τ has the following nice properties:

- $\tau(L) = L$
- $\mathrm{Cl}(V_-)'' \to (\mathrm{Cl}(V_+)'')^{\mathsf{op}}, a \mapsto \Lambda_\tau a^* \Lambda_\tau$ is an isomorphism.

Using the above isomorphism we turn the $\mathrm{Cl}(V_-)''$-$(\mathrm{Cl}(V_+)'')^{\mathsf{op}}$ bimodule $\mathcal{F}$ into a $\mathrm{Cl}(V_-)''$-$\mathrm{Cl}(V_-)''$ bimodule $\mathcal{F}_-$.

Lemma

$\mathcal{F}_- \cong L^2_\Omega(\mathrm{Cl}(V_-)'')$ for $\Omega = 1 \in \mathbb{C} \subset \mathcal{F}_-$ and hence $\mathcal{F}_- \boxtimes_{\mathrm{Cl}(V_-)''} \mathcal{F} \cong \mathcal{F}$

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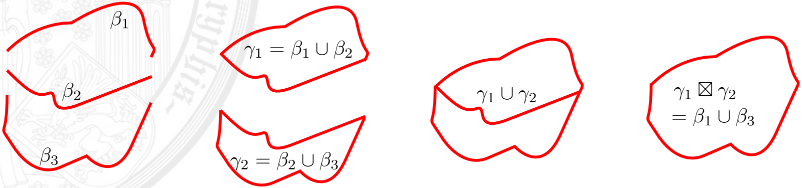
Smooth loop space of a manifold

Definition

The space of *smooth loops/paths with sitting instants* is

$$LM = \{\gamma \in C^\infty(S^1, M) \mid \gamma \text{ is locally constant at } 0 \text{ and } \pi\},$$
$$PM = \{\beta \in C^\infty([0,\pi], M) \mid \beta \text{ is locally constant at } 0 \text{ and } \pi\}.$$

LM and PM are diffeological spaces. Gluing is well defined:



Locality of $\mathcal{F}(LM)$

- $\mathcal{F}(LM)$ is a bundle over LM.
- But we started with a string structure on M.
- Does $\mathcal{F}(LM)$ see the underlying space M?

How do fibres over glueable loops relate?

Claim:

$$\mathcal{F}(LM)_{\gamma_1} \boxtimes \mathcal{F}(LM)_{\gamma_2} \xrightarrow{\simeq} \mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2}$$

$\boxtimes$ is Connes fusion.

Locality of $\mathrm{Cl}(LM)$

- $V_\pm := \{f \in V \mid \mathrm{supp}(f) \subseteq I_\pm\}$.
- $P\,\mathrm{Spin}(d) \circlearrowright \mathrm{Cl}(V_\pm)$.
- $P\,\mathrm{Spin}(M) \times_{P\,\mathrm{Spin}(d)} \mathrm{Cl}(V_\pm) =: \mathrm{Cl}_\pm(PM) \to PM$.

Locality of $\mathrm{Cl}(LM)$

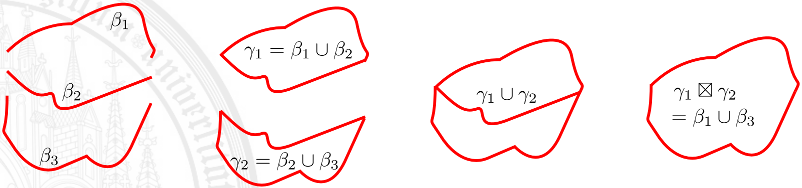
$\mathrm{Cl}_+(PM) \hat{\otimes}_{\mathrm{ev}} \mathrm{Cl}_-(PM) \simeq \mathrm{Cl}(LM)$.

First sign of locality of $\mathcal{F}(LM)$

Fix $\beta_1 \cup \beta_2 = \gamma \in LM$, for $\beta_1, \beta_2 \in PM$, then $\mathcal{F}(LM)_\gamma$ is a bimodule:

$$\mathrm{Cl}_+(PM)_{\beta_1} \circlearrowright \mathcal{F}(LM)_\gamma \circlearrowleft \mathrm{Cl}_-(PM)^{\mathrm{op}}_{\beta_2}.$$

Locality of $\mathcal{F}(LM)$



Fusion of Fock spaces

Goal: Construct, for each triple $(\beta_1, \beta_2, \beta_3)$, an isomorphism

$$\mathcal{F}(LM)_{\gamma_1} \boxtimes_{\mathrm{Cl}_\pm(PM)_{\beta_2}} \mathcal{F}(LM)_{\gamma_2} \xrightarrow{\simeq} \mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2}.$$

The construction should be natural.

Sketch of construction

- We have a natural isomorphism of the canonical fibre: $\mathcal{F} \boxtimes_{\mathrm{Cl}_\pm(V)} \mathcal{F} \xrightarrow{\simeq} \mathcal{F}$.
- Find a map μ

$$
\begin{array}{ccc}
\mathcal{F}(LM)_{\gamma_1} \boxtimes_{\mathrm{Cl}_\pm(PM)_{\beta_2}} \mathcal{F}(LM)_{\gamma_2} & & \mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2} \\
\big\downarrow (\varphi_1, \varphi_2) & \overset{\mu}{\longmapsto} & \varphi_3^{-1} \big\uparrow \\
\mathcal{F} \boxtimes \mathcal{F} & & \mathcal{F}
\end{array}
$$

Sketch of construction

- We have a natural isomorphism of the canonical fibre: $\mathcal{F} \boxtimes_{\mathrm{Cl}_\pm(V)} \mathcal{F} \xrightarrow{\simeq} \mathcal{F}$.
- Find a map μ, such that the top arrow in the diagram

$$
\begin{array}{ccc}
\mathcal{F}(LM)_{\gamma_1} \boxtimes_{\mathrm{Cl}_\pm(PM)_{\beta_2}} \mathcal{F}(LM)_{\gamma_2} & \longrightarrow & \mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2} \\
\Big\downarrow (\varphi_1, \varphi_2) & \overset{\mu}{\longmapsto} & \varphi_3^{-1} \Big\uparrow \\
\mathcal{F} \boxtimes \mathcal{F} & \longrightarrow & \mathcal{F}
\end{array}
$$

does not depend on the choice of (φ_1, φ_2)

Fusing trivializations

Given: $\beta_i \in PM$, $i = 1, 2, 3$.

$$\gamma_1 := \beta_1 \cup \beta_2 \qquad \gamma_2 := \beta_2 \cup \beta_3 \qquad \gamma_3 := \beta_1 \cup \beta_3$$

- Pick lifts $\widetilde{L\,\mathrm{Spin}}(M) \ni p_i \mapsto \gamma_i \in LM$, ($i = 1, 2$).
- Waldorf: There is a map $(p_1, p_2) \mapsto p_3 \ni \widetilde{L\,\mathrm{Spin}}(M)_{\gamma_3}$.
- Set $\mathcal{F}(LM)_{\gamma_i} \ni [p_i, v] \xmapsto{\varphi_i} v \in \mathcal{F}$.

Summary

- Described a representation $\widetilde{L\,\mathrm{Spin}(d)} \circlearrowright \mathcal{F}$.
- Given a manifold M, equipped with a string structure, constructed a vector bundle $\mathcal{F} \to LM$.
- Constructed a map $\mathcal{F}(LM)_{\gamma_1} \boxtimes_{\mathrm{Cl}_\pm(PM)_{\beta_2}} \mathcal{F}(LM)_{\gamma_2} \xrightarrow{\simeq} \mathcal{F}(LM)_{\gamma_1 \boxtimes \gamma_2}$ expressing that $\mathcal{F} \to LM$ is *local* in M.

Further work

- “Untransgress” the bundle $\mathcal{F} \to LM$ to a (2-vector?) bundle over M.
- The diffeomorphism group of the circle acts in LM. Lift this action to a bundle action $\mathcal{F} \to LM$.
- Equip $\mathcal{F} \to LM$ with a notion of parallel transport over surfaces.