Operator algebras: Exercises 5

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Read each exercise completely before you start, there are hints.

1 Causal complement in Minkowski spacetime

Let $S \subseteq \mathbb{R}^n$. Then we define $J^+(S)$ to be the future of S, and we define $J^-(S)$ to be the past of S. We say that $S \subseteq \mathbb{R}^n$ is *causally disjoint* from $S' \subseteq \mathbb{R}^n$, denoted $S \perp S'$, if

$$(J^+(S) \cup J^-(S)) \cap S' = \emptyset.$$

Prove that $\perp$ is a causal disjointness relation.

Let g be the Minkowski metric on $\mathbb{R}^n$. Show that the function

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$
$$(x, y) \mapsto g(x - y, y - x),$$

induces the causal complement

$$S^\perp := \mathbb{R}^n \setminus (J^+(S) \cup J^-(S)).$$

Prove that for every compact subset $A \subset \mathbb{R}^n$, the function

$$s_A : \mathbb{R}^n \to \mathbb{R},$$
$$y \mapsto \sup_{x \in A} f(x, y),$$

assumes negative values.

Write $D_{x,y} \subset \mathbb{R}^n$ for the open double cone generated by $x, y \in \mathbb{R}^n$. Prove that the pair $(\mathcal{I}, \perp)$, with $\mathcal{I}$ the collection of open bounded subsets of $\mathbb{R}^n$, is a causal index set, compatible with the topology. (Hint: Show that setting $\mathcal{I}_0$ to be the set of all open double cones $D_{x,y}$ does the trick.)

Show that the Poincaré group leaves the relation $\perp$ invariant.

2 Causal net index on an infinite set

Let $\mathcal{X}$ be an infinite set with a causal complement $\perp$, which additionally satisfies the following property:

There is a sequence $\{X_n\}_{n=1}^\infty$ of subsets $\emptyset \subsetneq X_n \subsetneq \mathcal{X}$, which are pairwise distinct, (i.e. $X_i \neq X_j$ for $i \neq j$), and with $X_n^\perp \neq \emptyset$ for all n and $\cup_{n=1}^\infty X_n = \mathcal{X}$.

Prove that the pair $(\mathcal{I}_0, \perp)$, where

$$\mathcal{I}_0 := \{M \subsetneq \mathcal{X} \mid M \neq \emptyset, M \subseteq X_n \text{ for some } n\},$$

is a causal index set.

3 A technical result

In the proof of the statement "ω is clustering if and only if $\mathcal{Z}^{\omega}_{\perp} = \mathbb{C}1$" (Lanford-Ruelle). We have the used the following technical result:

Let $\mathcal{N}_\kappa$ be a decreasing net of von Neumann algebras on a Hilbert space H and define $\mathcal{N} = \cap_\kappa \mathcal{N}_\kappa$. Assume that $\Omega \in H$ is cyclic for $\mathcal{N}'$, and that there exists a net of elements $N_\kappa \in \mathcal{N}_\kappa$ such that the following weak limits exist

$$u = \lim_\kappa N_\kappa \Omega \qquad u^* = \lim_\kappa N_\kappa^* \Omega.$$

Then $u, u^* \in \mathrm{clo}(\mathcal{N}\Omega)$.

Prove this statement, using the following steps:

- Define $w = (u + u^*)/2$ and prove that for all $a \in \mathcal{N}'$ we have $\langle aw, \Omega\rangle = \langle a\Omega, w\rangle$.
- Set $P \in \mathcal{N}'$ to be the projection onto $\mathrm{clo}(\mathcal{N}\Omega)$. Prove that $w - Pw$ is orthogonal to the set of vectors $\mathcal{N}'\Omega$.
- Conclude that $w - Pw = 0$. (Recall that Ω is cyclic for $\mathcal{N}'$.) And hence that $w = Pw \in \mathrm{clo}(\mathcal{N}\Omega)$.
- Argue similarly that
$$\frac{u - u^*}{2i} \in \mathrm{clo}(\mathcal{N}\Omega),$$
and finish the proof.

4 The C*-norm on $\mathrm{CAR}(H, \Gamma)$

Let H be a Hilbert space equipped with an anti-unitary involution Γ. In the lecture we claimed that for all $h \in H$ we have

$$\|A(h)\| = \frac{1}{\sqrt{2}} \left(\|h\|^2 + \left(\|h\|^4 - |\langle h, \Gamma h\rangle|^2 \right)^{1/2} \right)^{1/2}.$$

Prove this statement, following these steps:

- Prove that for all $h \in H$ we have
$$\begin{aligned} A(h)^2 &= \frac{1}{2}\langle h, \Gamma h\rangle 1, \\ (A(h)^*)^2 &= \frac{1}{2}\overline{\langle h, \Gamma h\rangle} 1. \end{aligned}$$
- Use this to prove that for all $h \in H$ we have
$$[A(h)^*, A(h)]_+^2 - [A(h)^*, A(h)]^2 = |\langle h, \Gamma h\rangle|^2.$$
- Conclude that
$$[A(h)^*, A(h)]^2 = \left(\|h\|^4 - |\langle h, \Gamma h\rangle|^2 \right) 1.$$
- From this, argue what the possibilities for the spectrum of $[A(h)^*, A(h)]$ are.

- If $[A(h)^*, A(h)]$ is a multiple of the identity, prove that it is zero.
- If $[A(h)^*, A(h)]$ is not a multiple of the identity, determine the spectrum of the self-adjoint operator $A(h)^* A(h)$ using what you know about the spectrum of $[A(h)^*, A(h)]$.
- Complete the proof.