Operator algebras: Exercises 4

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Read each exercise completely before you start, there are hints.

1 Basic opens of the strong topology

In the lecture we have defined the strong topology on $B(H)$. Write down the basic opens of the strong topology.

2 The double commutant theorem

Review the proof of Lemma 4.1.4 in Murphy. In it, we defined the following map

$$\varphi : B(H) \to B(H^{(n)}),$$
$$v \mapsto \begin{pmatrix} v & 0 & \dots & 0 \\ 0 & v & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & v \end{pmatrix}.$$

Prove that if A is a subalgebra of $B(H)$ and if $u \in A''$, then $\varphi(u) \in \varphi(A)''$.

3 Multiplication is not strongly continuous

Prove that if H is an infinite dimensional Hilbert space, then multiplication in $B(H)$ is not strongly continuous. (See exercise 4.3 in Murphy for an approach.)

Consider the subset

$$U(H) := \{v \in B(H) \mid v^*v = vv^* = 1\}.$$

Prove that multiplication $\mu : U(H) \times U(H) \to U(H)$ is strongly continuous.

Hint: Fix arbitrary $u \in U(H), f \in H$ and $\varepsilon > 0$ and consider the strong open neighbourhood

$$\mathcal{V} := \{x \in U(H) \mid \|(x-u)f\| < \varepsilon\}.$$

Let $(x_0, y_0) \in \mu^{-1}(\mathcal{V})$ be arbitrary, and show that for appropriate $\varepsilon' > 0$ the strong open neighbourhood of (x_0, y_0) given by

$$\mathcal{W} := \{(x,y) \in U(H) \times U(H) \mid \|(x-x_0)y_0 f\| < \varepsilon', \|(y-y_0)f\| < \varepsilon'\}$$

is contained in $\mu^{-1}(\mathcal{V})$.

Where does this argument break down if the operators involved are not unitary?

4 GNS representation

Do exercise 2 in Chapter 3 of Murphy, (page 107).

5 Density operators

Let H be a Hilbert space, and let $\rho \in B(H)^+$ be a density operator. (That is, the quantity ρu is trace-class for all $u \in B(H)$.) Prove that the assignment

$$\begin{aligned} \tau_\rho : B(H) &\to \mathbb{C}, \\ u &\mapsto \frac{\operatorname{tr}(\rho u)}{\operatorname{tr}(\rho)}, \end{aligned}$$

is a state on $B(H)$.

6 Fun with positive linear functionals

Let τ be a positive linear functional on a C*-algebra A. Prove that for all self-adjoint elements $a, b \in A$ we have

$$\tau(a^2)\tau(b^2) \geqslant |\tau(ab)|^2.$$

Hint: First prove that for all $\lambda \in \mathbb{C}$, we have

$$|\lambda^2|\tau(a^2) + 2\operatorname{Re}(\lambda\tau(ba)) + \tau(b^2) \geqslant 0.$$

Then, by choosing λ in a clever way, show that this implies that for all $t \in \mathbb{R}$ we have

$$t^2\tau(a^2) + 2t|\tau(ba)| + \tau(b^2) \geqslant 0.$$

Then, view this as a quadratic equation in t.

7 Pure states on $C_0(\mathbb{R})$

Following the steps below, prove that the pure states on the C*-algebra $C_0(\mathbb{R})$ are exactly the point measures:

$$\tau_x(f) = f(x), \quad x \in \mathbb{R},\ f \in C_0(\mathbb{R}).$$

First, argue using the Riesz-Markov-Kakutani theorem for the continuous dual of $C_0(\mathbb{R})$ that any state on $C_0(\mathbb{R})$ is given by some probability measure μ on $\mathbb{R}$. (A probability measure on $\mathbb{R}$ is a measure μ on $\mathbb{R}$ such that $\mu(U) \geqslant 0$ for all $U \subseteq \mathbb{R}$ and such that $\mu(\mathbb{R}) = 1$.)

Then, suppose that we have probability measures μ_1 and μ_2 and a $t \in (0, 1)$ such that $t\mu_1 + (1 - t)\mu_2 = \delta_x$. Evaluate the measures on the singleton $\{x\}$, what can we conclude?