Operator algebras: Exercises 2

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1 Chapter 1, Exercise 6

Do exercise 6 from Chapter 1 in Murphy.

2 The exponential is in general not surjective onto $\mathrm{Inv}(A)$

Find a compact topological space Ω, and an invertible element $f \in \mathrm{Inv}(C(\Omega))$ such that there does not exist an element $g \in C(\Omega)$ with the property that $e^g = f$.

3 The spectrum is closed in the weak* topology

Let A be an Abelian Banach algebra. Prove that $\Omega(A) \cup \{0\}$ is closed in the weak* topology.

4 The Gelfand transform is continuous

Let A be an Abelian Banach algebra. Prove that, for each $a \in A$, the Gelfand transform

$$\begin{aligned} \hat{a} : \Omega(A) &\to \mathbb{C}, \\ \tau &\mapsto \tau(a), \end{aligned}$$

is continuous with respect to the weak* topology.

5 The unitization of a C*-algebra

Let A be a non-unital C*-algebra. Denote its unitization by $\tilde{A}$. For each $x \in \tilde{A}$ we write

$$\begin{aligned} L_x : A &\to A, \\ a &\mapsto xa. \end{aligned}$$

Prove that the map

$$\begin{aligned} \tilde{A} &\to \mathbb{R}, \\ x &\mapsto \|L_x\|, \end{aligned}$$

is a norm on $\tilde{A}$ that turns $\tilde{A}$ into a C*-algebra.

6 Finite dimensional Abelian C*-algebras

Suppose that A is a non-zero finite dimensional Abelian C*-algebra. Prove that $A \simeq \mathbb{C}^n$ for some $n \in \mathbb{N}$. (Here we equip $\mathbb{C}^n$ with entry-wise operations, what is the, necessarily unique, norm that turns this into a C*-algebra?)

Conclude that any finite dimensional Abelian C*-algebra is unital.