Operator algebras: Exercises 1

October 18, 2018

1 Continuous maps

Let $(\mathbb{R}, \mathcal{O})$ be the topological space $\mathbb{R}$, with the topology generated by the sets

$$B_\varepsilon(x) := \{y \in \mathbb{R} \mid |x - y| < \varepsilon\},$$

for $x \in \mathbb{R}$ and $\varepsilon > 0$. Recall that a function $f : (\Omega_1, \mathcal{O}_1) \to (\Omega_2, \mathcal{O}_2)$ is *continuous* if for all $U \in \mathcal{O}_2$ we have $f^{-1}(U) \in \mathcal{O}_1$.

1. Prove that a function $f : (\mathbb{R}, \mathcal{O}) \to (\mathbb{R}, \mathcal{O})$ is continuous in this sense if and only if it is continuous in the ε-δ sense.
2. Give the two analogous definitions of continuity for maps $f : V \to W$ between Banach spaces, and show that these definitions are equivalent.

2 Compact Spaces

State the Heine-Borel theorem.

3 Banach Algebras

1. Let Ω be a compact topological space. Prove that

   $$C(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ is continuous}\},$$

   equipped with the sup-norm is a Banach algebra. (Bonus: What goes wrong if Ω is not compact?)
2. Let $\mathcal{H}$ be a Hilbert space. Prove that

   $$\mathcal{B}(\mathcal{H}) := \{T : \mathcal{H} \to \mathcal{H} \mid T \text{ is bounded and linear}\},$$

   equipped with the operator norm is a Banach algebra.

4 Quotient by an ideal

Complete the proof of Theorem 1.1.1 in Murphy, more specifically, prove that the expression $\|a + I\| = \inf_{b \in I} \|a + b\|$ defines a norm on A/I.

5 Beurling/Gelfand

In the proof of Beurling's theorem we used the following statement. Let A be a unital Banach algebra, and let $a \in A$. Then the function $f : \mathbb{C} \setminus \sigma(a) \to \mathbb{C}, a \mapsto (a - \lambda)^{-1}$ has a series expansion, around ∞,

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$$

which converges for all $\lambda \in \mathbb{C}$ with $|\lambda| > r(a)$. Prove this statement. *Hint: Use that we have already shown that f is holomorphic for $|\lambda| > r(a)$, and some appropriate result from complex analysis.*

6 Direct sums of Banach algebras

Do exercise 1 in Chapter 1 of Murphy.